Partial Variational Principle for Electromagnetic Field Problems: Theory and Applications

SHYH-JONG CHUNG AND CHUN HSIUNG CHEN

Abstract —A partial variation concept is proposed to clarify and extend the ideas and techniques used in variational electromagnetics (VEM) and variational reaction theory (VRT) of recent papers. Based on this concept, a partial variational principle (PVP) is established for handling a general linear time-harmonic interior and/or exterior electromagnetic field problem. This principle is then applied to attack the problem of waves incident on a dielectric discontinuity in a parallel-plate guide. Also included are numerical results and discussions about such waveguide discontinuity problems for illustrating the use of the proposed technique.

I. Introduction

VARIATIONAL TECHNIQUES have been applied extensively and successfully in establishing stationary formulas for some physical quantities, e.g., resonant frequencies, cutoff frequencies, antenna impedances, and scattering cross sections [1]. Due to its effectiveness in handling problems with curved boundaries and/or with inhomogeneous and anisotropic materials, the variational formulation coupled with the finite element method has become a standard technique in studying many interior electromagnetic field problems [2]–[9].

Recent efforts by several investigators also extend the scope of variational techniques to the regime of more difficult exterior problems, such as radiation, scattering, and dielectric waveguide problems, where the radiation and continuity conditions should be properly incorporated [10]–[19]. Silvester and Hsieh [10] divided the entire region into interior and exterior ones, and applied Green's theorem to obtain a variational equation by treating the exterior region as an exterior element. McDonald and Wexler [11], on the other hand, used an integral equation as a constraint on the variational equation to replace the exterior element. Mei proposed the unimoment method [12], in which he imposed an artificial boundary and expressed the exterior and interior fields as sums of eigenmodes and pseudomodes, respectively. The coefficients of both series were then obtained by matching the continuity conditions on the artificial boundary. Jeng and Chen [15], [16] tried to properly handle the radiation and continuity conditions by imposing suitable constraints on the fundamental variational principle (FVP) derived in the variational electromagnetics (VEM). Having primarily based their works on the induction theorem, reciprocity theorem, and reaction concept, Jeng and Chen's theory may need some simplification. Recently, Wu and Chen [17] developed a simpler variational reaction theory (VRT), which only makes use of the concepts of reaction and test fields. They then applied it to the exterior problems of dielectric waveguides and scattering [17], [18]. In VRT, the radiation and continuity conditions were also properly absorbed into the variational formulation by enforcing suitable constraints on the trial sources.

In the work on VRT, the notion of partial variation has actually been included implicitly in the formulation and the solution of the variational equation. Although the partial variation concept was later suggested in a paper by Jeng *et al.* [19], its use in applied problems still needs a thorough and systematic investigation. This paper tries to combine the idea of partial variation and those of VEM and VRT, with some extension, so that a "partial variational principle" may be established for dealing with a general linear electromagnetic field problem. In this study, this principle will be applied to attack the dielectric discontinuity problem in a parallel-plate conducting waveguide, with several numerical results included for demonstrating the use of the principle.

II. Partial Variational Principle

The concepts of variation and partial variation in this paper are very similar to those of differentiation and partial differentiation in calculus. The major difference is that the differentiation is operated on a function and with respect to some variables, while the variation is operated on a functional (a function of function) and with respect to the so-called trial fields. In calculus, if g is a function of x and y, then its partial differentiation with respect to x is defined as the operation on g by fixing y. Similarly, in our subsequent discussion, the designated functional I is a functional of trial fields f and f^a. The partial variation of I with respect to f^a is thus defined as the operation on I by fixing f.

Let $S_0(\bar{\bar{\epsilon}}, \bar{\bar{\mu}}; s_0; f_0)$ be a time-harmonic linear electromagnetic field problem (called the original system) to be

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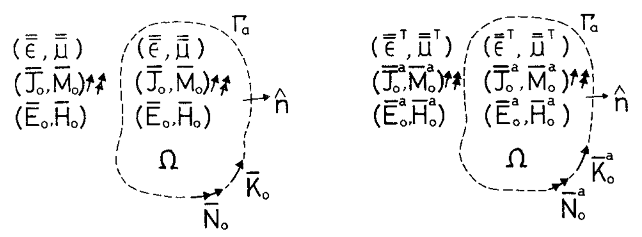

(a) (b)

Fig. 1 Geometry of electromagnetic field problem. (a) Original system $S_0(\bar{\bar{\epsilon}}, \bar{\bar{\mu}}; s_0, f_0)$. (b) Adjoint system $S_0^a(\bar{\bar{\epsilon}}^T, \bar{\bar{\mu}}^T; s_0^a; f_0^a)$.

solved (Fig. 1(a)). For a solution of S_0 by the variational formulation, we have to introduce a trial system (the original trial system), $S(\bar{\bar{\epsilon}}, \bar{\bar{\mu}}; s; f)$. Besides, the adjoint system $S_0^a(\bar{\bar{\epsilon}}^T, \bar{\bar{\mu}}^T; s_0^a; f_0^a)$ (Fig. 1(b)) of the original one S_0 and its trial system (the adjoint trial system) $S^a(\bar{\bar{\epsilon}}^T, \bar{\bar{\mu}}^T; s^a; f^a)$ are sometimes needed. Here, the superscript T in $\bar{\bar{\epsilon}}^T$ and $\bar{\bar{\mu}}^T$ means the transpose of the permittivity tensor $\bar{\bar{\epsilon}}$ and the permeability tensor $\bar{\bar{\mu}}$. The source s_β^α includes the volume sources $\bar{J}_\beta^\alpha$ (electric), $\bar{M}_\beta^\alpha$ (magnetic) as well as the surface sources $\bar{K}_\beta^\alpha$ (electric), $\bar{N}_\beta^\alpha$ (magnetic) over some surface Γ_a. The symbol f_β^α represents the fields $\bar{E}_\beta^\alpha$, $\bar{H}_\beta^\alpha$. The superscript α is a to denote the one for the adjoint and its trial systems; the absence of a superscript denotes the one for the original and its trial systems. The subscript β is 0 to denote the true one; the absence of a subscript denotes the trial one.

The reaction concept [1], [20] is useful in relating two systems. Let $s_1(s_2)$ be the sources of the first (second) system, which include the volume sources $\bar{J}_1(\bar{J}_2)$, $\bar{M}_1(\bar{M}_2)$ and the surface sources $\bar{K}_1(\bar{K}_2)$, $\bar{N}_1(\bar{N}_2)$. Let $f_1(f_2)$ be the fields of the first (second) system, which denote the electric field $\bar{E}_1(\bar{E}_2)$ and the magnetic field $\bar{H}_1(\bar{H}_2)$. The reaction of these two systems may be defined as (referring to Fig. 1):

$$\langle s_1|f_2\rangle = \langle f_2|s_1\rangle = \int_{\Omega_\infty}\left[\bar{E}_2\cdot\bar{J}_1 - \bar{H}_2\cdot\bar{M}_1\right]dv + \int_{\Gamma_a}\left[\bar{E}_2(\Gamma_a^p)\cdot\bar{K}_1 - \bar{H}_2(\Gamma_a^q)\cdot\bar{N}_1\right]ds. \quad (1)$$

Here the volume integral is extended over the whole infinite space Ω_∞, and the surface integral is over the surface Γ_a where the surface sources exist. The superscripts p, q associated with Γ_a may be $+$ or $-$, with $+$ denoting the outer side and $-$ the inner side of the boundary Γ_a. Essentially, (p, q) may be any one of the four combinations $(+,+)$, $(-,-)$, $(+,-)$, $(-,+)$. However to preserve the reciprocity theorem (that is, $\langle f_2|s_1\rangle = \langle f_1|s_2\rangle$), the signs of p, q in $\langle f_2|s_1\rangle$ and $\langle f_1|s_2\rangle$ should be properly chosen. If (p, q) in $\langle f_2|s_1\rangle$ is $(+,+)$, $(-,-)$, $(+,-)$, or $(-,+)$, then that of $\langle f_1|s_2\rangle$ should be chosen as $(-,-)$, $(+,+)$, $(+,-)$, or $(-,+)$, respectively. In addition, if $(-,-)$ or $(+,+)$ has been chosen, at least one of the trial surface sources $\bar{K}$ (or $\bar{K}^a$) and $\bar{N}$ (or $\bar{N}^a$) should be constrained to equal the true one.

In this study, the partial and total variational operators δ, δ^a, and δ^t are defined as

δ = the partial variational operator which operates only on the original trial system $S(s; f)$,

δ^a = the partial variational operator which operates only on the adjoint trial system $S^a(s^a; f^a)$;

$\delta^t = \delta + \delta^a$

= the total variational operator which operates on both the original trial system $S(s; f)$ and the adjoint trial system $S^a(s^a; f^a)$.

With these definitions, one immediately has the following equations:

$$\begin{aligned} \delta^a f &= 0 \\ \delta f^a &= 0 \\ \delta^t f &= \delta f + \delta^a f = \delta f \\ \delta^t f^a &= \delta f^a + \delta^a f^a = \delta^a f^a. \end{aligned} \quad (2)$$

They will also be true with f and f^a replaced by s and s^a, respectively.

We now establish three functionals for the partial variational principle. By the uniqueness theorem [1], the original trial fields f will be equal to the original true fields f_0 when the trial sources s are equal to the original true sources s_0. Therefore, the reaction of arbitrary adjoint test fields $\delta^a f^a$ on the difference of the trial sources s and the original true sources s_0 should vanish whenever the trial fields f are equal to the original true fields f_0 [17], i.e.,

$$\langle \delta^a f^a | s - s_0\rangle = 0. \quad (3)$$

By noting that the operator δ^a only operates on the adjoint quantities, it can thus be taken out of the reaction symbol,

$$\langle \delta^a f^a | s - s_0\rangle = \delta^a\langle f^a | s - s_0\rangle = 0. \quad (4)$$

With this, one may get a partial variational equation formulation of the original system as proposed by the variational reaction theory [17]:

$$\begin{cases} \delta^a I^a = 0 \\ I^a = \langle f^a | s - s_0\rangle. \end{cases} \quad (5)$$

In that theory, the concept of partial variation has already been included implicitly.

Similarly, when the adjoint trial fields equal the adjoint true fields, the reaction $\langle s^a - s_0^a | \delta f\rangle$ should be zero for arbitrary test fields δf. Thus one may obtain a partial variational equation formulation of the adjoint system as follows:

$$\begin{cases} \delta I = 0 \\ I = \langle s^a - s_0^a | f\rangle. \end{cases} \quad (6)$$

The partial variational equation (5) or (6) has the solution either for the original system or for the adjoint system. If the solutions for these two systems are required at the same time, the preceding equations should be solved simultaneously. Of course, they can also be obtained through the application of the third variational equation discussed below.

The reaction of arbitrary test fields $\delta^t f^a$ upon the source difference $s - s_0$, and that of arbitrary test sources $\delta^t s$ on the field difference $f^a - f_0^a$, should add to zero whenever the original and adjoint trial systems equal their corresponding original and adjoint systems, i.e.,

$$\langle \delta^t f^a | s - s_0 \rangle + \langle f^a - f_0^a | \delta^t s \rangle = 0. \qquad (7)$$

By noting that

$$\langle \delta^t f^a | s - s_0 \rangle = \delta^t \langle f^a | s - s_0 \rangle - \langle f^a | \delta^t s \rangle \qquad (8)$$

One may cast (7) into a variational formulation as follows:

$$\begin{cases} \delta^t I^t = 0 \\ I^t = \langle f^a | s \rangle - \langle f^a | s_0 \rangle - \langle f_0^a | s \rangle. \end{cases} \qquad (9)$$

In general, the adjoint fields f_0^a are undetermined; thus it is not convenient to solve (9) directly. But if all sources are confined in a finite region, as was achieved by the use of the induction theorem [1], the reciprocity theorem will be satisfied. Then the term $\langle f_0^a | s \rangle$ in (9) may be replaced by $\langle s_0^a | f \rangle$ so that one may obtain the fundamental variational principle as suggested by [16]:

$$\begin{cases} \delta^t I^t = 0 \\ I^t = \langle f^a | s \rangle - \langle f^a | s_0 \rangle - \langle s_0^a | f \rangle. \end{cases} \qquad (10)$$

This variational equation (10) certainly will give the solutions to the original as well as the adjoint systems. Note that the variational formulations, (5), (6), (9), or, (10), may be related by noting the expressions in (2) and (7), i.e.,

$$\begin{aligned} \delta^t I^t &= \langle \delta^a f^a | s - s_0 \rangle + \langle f^a - f_0^a | \delta s \rangle \\ &= \delta^a \langle f^a | s - s_0 \rangle + \delta \langle f^a - f_0^a | s \rangle \\ &= \delta^a I^a + \delta I \end{aligned} \qquad (11)$$

where the preservation of the reciprocity theorem ($\langle f^a - f_0^a | s \rangle = \langle s^a - s_0^a | f \rangle$) has been assumed. The relation between FVP and VRT is now made clear. The total variation operation on the functional I^t in FVP can thus be divided into two parts; one is the operation in VRT which partially acts on the functional I^a and the other partially operates on the functional I.

III. Discontinuity in Dielectric-Filled Parallel-Plate Guide

In this section, we try to apply the partial variational principle, together with the finite element method and frontal solution technique, to solve the discontinuity problems due to partial dielectric filling in a parallel-plate waveguide.

The geometry of the dielectric-filled waveguide is shown in Fig. 2(a), where the distance between two conducting plates is d. Here placed in region I ($-\infty < z < 0$) is a dielectric slab of thickness t_1 and refractive index n_1, and in region II ($l < z < \infty$) is another dielectric slab of thickness t_2 and refractive index n_2. Between these two slabs is the region III ($0 < z < l$), which has a completely arbitrary transition in shape and contains an inhomogeneous material. Let the TE_1 mode of guide I be incident from the left-hand side. Then some higher order modes would be excited and reflected back to guide I, and some modes of guide II would also be excited and propagated into the right-hand side.

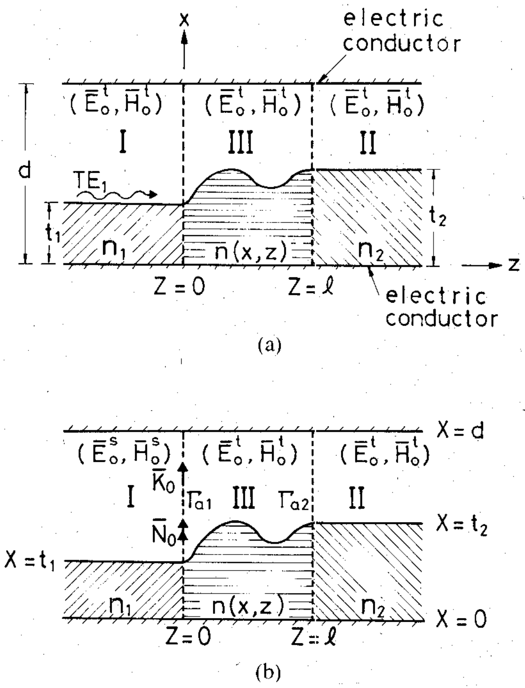

Fig. 2. Geometry of partially dielectric-filled discontinuity problem. (a) Original problem with TE_1 mode incident. (b) Induction equivalent of (a).

For convenience, we consider the related problem of Fig. 2(b), where we keep the same total fields $\overline{E}_0^t$, $\overline{H}_0^t$ in regions II and III, but excite the scattered fields $\overline{E}_0^s = \overline{E}_0^t - \overline{E}_0^i$, $\overline{H}_0^s = \overline{H}_0^t - \overline{H}_0^i$ in region I. Here $\overline{E}_0^i$, $\overline{H}_0^i$ are the incident fields and $\overline{K}_0$, $\overline{N}_0$ are the surface electric and magnetic sources such that

$$\overline{K}_0(\Gamma_{a1}) = \hat{n} \times [\overline{H}_0^s - \overline{H}_0^t]|_{\Gamma_{a1}} = \hat{y} H_{0x}^i|_{\Gamma_{a1}} \qquad (12)$$

$$\overline{N}_0(\Gamma_{a1}) = -\hat{n} \times [\overline{E}_0^s - \overline{E}_0^t]|_{\Gamma_{a1}} = \hat{x} E_{0y}^i|_{\Gamma_{a1}}. \qquad (13)$$

After transforming the problem of Fig. 2(a) to that of Fig. 2(b), the new problem will preserve the reciprocity theorem. Since only the solutions of the original problem are of interest, we start with the partial variational formulation (5). From (5) and (1), one has

$$\begin{aligned} I^a = &\int_{\mathrm{I+II+III}} [\overline{E}^a \cdot (\overline{J} - \overline{J}_0) - \overline{H}^a \cdot (\overline{M} - \overline{M}_0)]\, dv \\ &+ \int_{\Gamma_{a1}} [\overline{E}^a(\Gamma_{a1}^-) \cdot (\overline{K} - \overline{K}_0) - \overline{H}^a(\Gamma_{a1}^+) \cdot (\overline{N} - \overline{N}_0)]\, ds \\ &+ \int_{\Gamma_{a2}} [\overline{E}^a(\Gamma_{a2}^-) \cdot \overline{K} - \overline{H}^a(\Gamma_{a2}^+) \cdot \overline{N}]\, ds. \end{aligned} \qquad (14)$$

Here (p, q) in (1) has been chosen as $(-, +)$. $\overline{J}_0(\overline{K}_0)$ and $\overline{M}_0(\overline{N}_0)$ are the true volume (surface) electric and magnetic sources, respectively, while $\overline{E}^a$ and $\overline{H}^a$ are the adjoint trial fields. The relations between the trial sources $\overline{J}$, $\overline{K}$, $\overline{M}$, $\overline{N}$, and the trial fields $\bar{E}, \bar{H}$ are given by

$$\bar{M} = -\nabla \times \bar{E} - j\omega\bar{\bar{\mu}}\cdot\bar{H} \tag{15}$$

$$\bar{J} = \nabla \times \bar{H} - j\omega\bar{\bar{\epsilon}}\cdot\bar{E} \tag{16}$$

$$\bar{K} = \hat{n} \times \left[\bar{H}(\Gamma_a^+) - \bar{H}(\Gamma_a^-)\right] \tag{17}$$

$$\bar{N} = -\hat{n} \times \left[\bar{E}(\Gamma_a^+) - \bar{E}(\Gamma_a^-)\right]. \tag{18}$$

Here $\bar{\bar{\mu}} = \mu_0\bar{\bar{I}}$, $\bar{\bar{\epsilon}} = n^2(x,z)\epsilon_0\bar{\bar{I}}$. $\bar{\bar{I}}$ is the unit dyad.

To make the functional in (14) numerically solvable, we add the following constraints:

i) $\bar{J} = \bar{J}_0(=0)$, $\bar{M} = \bar{M}_0(=0)$ in regions I, II;
ii) $\bar{M} = \bar{M}_0(=0)$ in region III.

By constraint i), the integration region of (14) is now confined to region III, and by ii) the magnetic field $\bar{H}$ in region III can be expressed in terms of $\bar{E}$:

$$\bar{H} = \frac{-1}{j\omega}\bar{\bar{\mu}}^{-1}\cdot\nabla \times \bar{E}. \tag{19}$$

Up to this stage, the unknown variables can be divided into three groups. The first ones are the trial fields in region I, which must satisfy the source free condition (according to constraint i)) and the boundary conditions at $x = 0$ and $x = d$. Thus, they can be written in terms of the left-going modes of guide I, that is,

$$\bar{E}^s(\mathrm{I}) = \sum_l R_l\bar{e}_l^{\mathrm{I}}(x)\cdot\exp\left(j\beta_l^{\mathrm{I}}z\right)$$

$$\bar{H}^s(\mathrm{I}) = \sum_l R_l\bar{h}_l^{\mathrm{I}}(x)\cdot\exp\left(j\beta_l^{\mathrm{I}}z\right). \tag{20}$$

The second ones are the trial fields in region II, which may be expanded into the right-going modes of guide II:

$$\bar{E}^t(\mathrm{II}) = \sum_l T_l\bar{e}_l^{\mathrm{II}}(x)\cdot\exp\left[-j\beta_l^{\mathrm{II}}(z-l)\right]$$

$$\bar{H}^t(\mathrm{II}) = \sum_l T_l\bar{h}_l^{\mathrm{II}}(x)\cdot\exp\left[-j\beta_l^{\mathrm{II}}(z-l)\right]. \tag{21}$$

Here $\bar{e}_l^{\mathrm{I}}(\bar{e}_l^{\mathrm{II}})$, $\bar{h}_l^{\mathrm{I}}(\bar{h}_l^{\mathrm{II}})$ are the normalized mode functions in guide I (II) with propagation constants $\beta_l^{\mathrm{I}}(\beta_l^{\mathrm{II}})$, and $R_l(T_l)$ are the coefficients to be determined. Physically, R_1 and T_1 are the reflection and transmission coefficients of this problem, respectively. The third ones are the trial fields in region III, and they must satisfy the boundary conditions at $x = 0$ and $x = d$ and may be expanded by the local bases of the finite element method.

Without further constraints, these three groups of trial fields will be treated independently. The discontinuity in the trial fields between each region is then supported by the trial surface sources $\bar{K}, \bar{N}$. Thus, as the finite element method and the frontal solution technique are used, we may number the regions I, II and the nodes in Γ_{a1} and Γ_{a2} as the last element. After the assembly and elimination process in the frontal solution technique, the working matrix of the front will contain the unknown coefficients of the modes of guides I, II and the unknown nodal field values along Γ_{a1} and Γ_{a2} only. Hence these unknown modal coefficients and nodal field values can be directly obtained without any back-substitution process.

In the following study, we add one more constraint:

iii) $\bar{K} = \bar{K}_0$, $\bar{N} = \bar{N}_0$ at boundary Γ_{a1};
$\bar{K} = \bar{N} = 0$ at boundary Γ_{a2}.

Thus the unknown variables can further be reduced.

With this constraint in (14), $\bar{K} - \bar{K}_0$, $\bar{N} - \bar{N}_0$ on Γ_{a1} and $\bar{K}$, $\bar{N}$ on Γ_{a2} become zero. The remaining volume integral, after replacing $\bar{J}, \bar{M}$ by (15), (16) and using integration by parts, becomes a volume integral of the fields in region III plus a boundary integral of the fields just inside the boundary Γ_a. With the same constraint, $\bar{H}(\Gamma_a^-)$ in this boundary integration must be expressed by $\bar{H}(\Gamma_a^+)$. By imposing all the constraints,

$$I^a = \frac{j}{\omega\mu_0}\int_{\mathrm{III}}\left[\frac{\partial E_y^a}{\partial x}\cdot\frac{\partial E_y}{\partial x} + \frac{\partial E_y^a}{\partial z}\cdot\frac{\partial E_y}{\partial z} - k_0^2n^2E_y^aE_y\right]dv$$
$$-\int_{\Gamma_{a1}} E_y^a(0^+)\cdot\left[H_x(0^-) + H_{0x}^i(0)\right]dx$$
$$-\int_{\Gamma_{a2}} E_y^a(l^-)\cdot H_x(l^+)\,dx \tag{22}$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$, and H_{0x}^i is the x component of the incident magnetic field. Note that, by constraint iii) and the orthogonal properties of the modes in guides I and II, $H_x(0^-)$ and $H_x(l^+)$ (which are expanded in terms of modes of guides I and II) can be related to $E_y(0^+)$ and $E_y(l^-)$ (which are expressed in terms of nodal fields along boundaries Γ_{a1} and Γ_{a2}), respectively.

With (5) and (22), and employing the finite element method coupled with the frontal solution technique, we may obtain the fields on the boundaries Γ_{a1} and Γ_{a2}. Then by constraint iii), the modal coefficients can be solved immediately.

IV. Finite Element Method and Numerical Results

The partial variational equation (22) will be solved by the finite element method [21] together with the frontal solution technique [22]. In this study, the second-order triangular elements, each with six nodes (as shown in Fig. 3(a)), will be used. The shape functions N_1 to N_6 are given by

$$N_1 = L_1(2L_1 - 1) \qquad N_2 = L_2(2L_2 - 1)$$
$$N_3 = L_3(2L_3 - 1) \qquad N_4 = 4L_1L_2$$
$$N_5 = 4L_2L_3 \qquad N_6 = 4L_3L_1 \tag{23}$$

where L_1, L_2, L_3 are the area coordinates. The relation between the area coordinates and Cartesian coordinates is given by

$$\begin{bmatrix} z \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

where (x_j, z_j) are the Cartesian coordinates of the jth vertex of the triangle ($j = 1,2,3$).

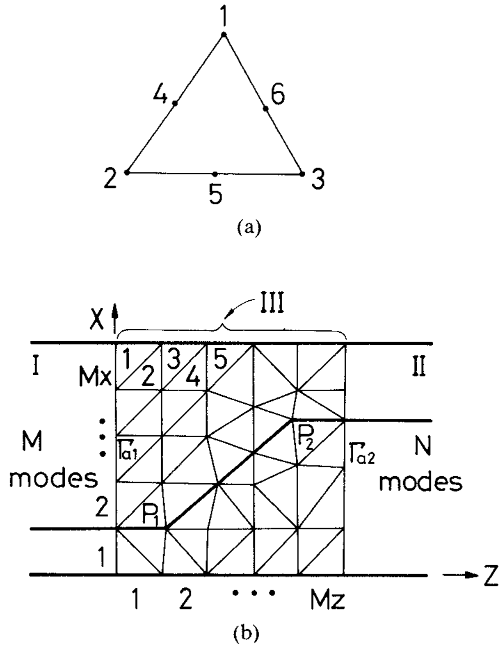

Fig. 3. (a) Second-order triangular element. (b) Typical mesh division for tapered-type dielectric discontinuity in parallel-plate guide. The fields in guide I and guide II are expanded by M and N local modes, respectively.

Typical meshes for a tapered discontinuity problem are shown in Fig. 3(b). There are M_x intervals along the x axis and M_z intervals along the z axis. The fields in the left- and right-hand sides of the finite element region are expanded into M and N local waveguide modes, respectively. When dealing with the frontal solution technique, the sequence in numbering the elements should be along the direction of the z axis in order to minimize the dimension of the working matrix. Another point worthy of notice is that the boundaries Γ_{a1} and Γ_{a2} should not be chosen to cross the points p_1 and p_2 of Fig. 3(b). By this choice, it is thus possible to divide the meshes even when the tapered angle is almost (but not exact) 90°. Besides, since the matching boundaries (Γ_{a1} and Γ_{a2}) are away from the discontinuity region (the region between p_1 and p_2), the mode number needed to express the fields in guides I and II may be reduced because of the decaying nature of the evanescent modes.

The phenomenon of the relative convergence, discussed in other numerical works [23]–[25], is also investigated. Fig. 4 shows the field distribution $E_y(z=0^-)$ for a step dielectric discontinuity in a parallel-plate guide. Here, we use $M_x=10$, $M_z=2$, and $N=10$ (it is sufficient to expand the fields in guide II). There are 19 ($=2M_x+1-2$) free nodes in the x direction. When the mode number M varies from 10 to 16, the fields all behave the same and are proved to be the same as those at the immediate right-hand side of the junction. But as M is further increased, the coefficients of the higher order modes will be larger than the actual values. Consequently, the error reflected by the rapid variation would appear. This unwanted phenomenon may be explained as follows: As the mode number increases, the field distributions of the higher order modes

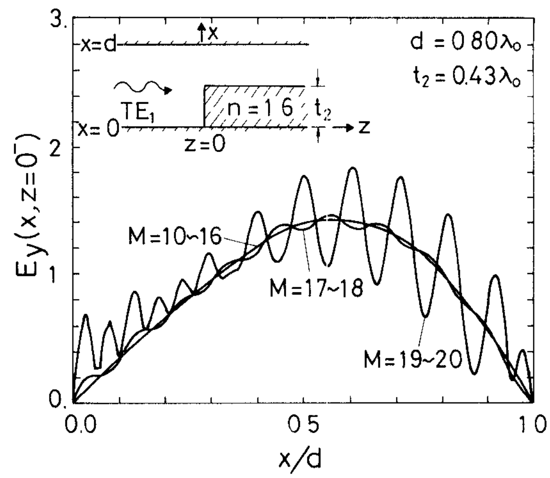

Fig. 4. Electric field distributions at the left-hand side of the junction to show relative convergence phenomena $M_x=10$, $M_z=2$, $N=10$.

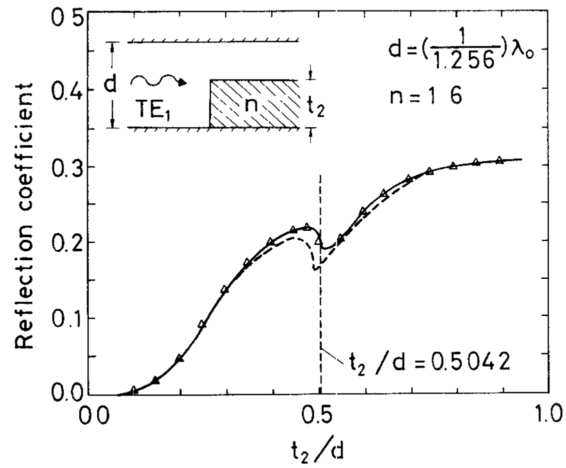

Fig. 5. Reflection coefficient for step-discontinuity problem ——: PVP, ΔΔΔ: mode-matching method, and ---: MRCT.

would not be adequately represented by the boundary fields with the small number (now 19) of nodes. Although the field distribution at the junction is highly dependent on the choice of the mode number and the node number in the transverse direction, the reflection coefficient is not affected by that choice. One possible explanation is that the fields of the higher order modes are integrated to zero in calculating the reflection coefficient.

As a check of the present method, we consider the step-type discontinuity (shown in Fig. 5) which can be tackled by other methods. Fig. 5 shows the variation in the reflection coefficient due to the variation in the dielectric width of guide II. At $t_2/d=0.5042$, the second mode will propagate in the partially loaded guide. It is shown that the curve of the present method and that of the mode-matching method match very well, and that the results of MRCT (the modified residue-calculus technique) [23] are almost the same as those of the above two methods except at the near neighborhood of $t_2/d=0.5042$.

In Fig. 6, a straight tapered dielectric discontinuity in the parallel-plate guide is considered. The dielectric slab in the partially loaded guide has a thickness of 0.43 λ_0 (λ_0

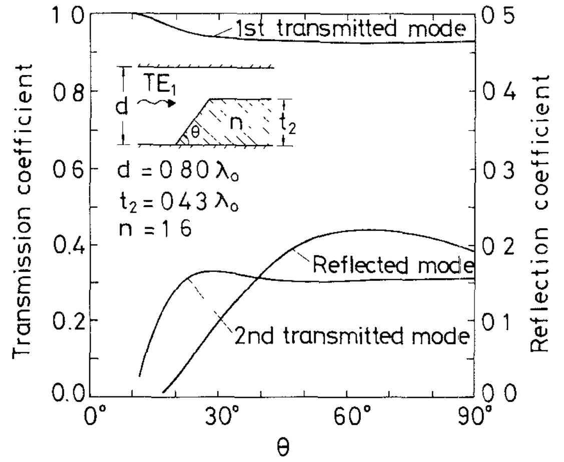

Fig. 6 Reflection and transmission coefficients for straight tapered-discontinuity problem.

being the wavelength in free space), and there are two guided modes in this guide. The three curves in the figure show the variations of the reflection coefficient in the dielectric-free guide (I) as well as the transmission coefficients of the first and second modes in the partially loaded guide (II) with respect to the change in the tapered angle. (Note that for each angle, the sum of the squares of the corresponding three values must equal unity because the power conservation law should be satisfied.) In the range from 10° to 45°, the three curves vary quickly, whereas outside this range, they change gently. It is noted that the reflection coefficient curve reaches a maximum at the tapered angle of 64° instead of 90°.

Fig. 7 shows the reflection coefficient and the amplitude coefficients of the two guided modes in the right-hand-side guide (guide II) as a function of t_2/d with various tapered angles as parameters. In Fig. 7(b), we show only the results for $t_2/d > 0.5042$. For t_2/d smaller than that value, the coefficients are zero for the second guided mode and are almost unity for the first mode of guide II. In the neighborhood of $t_2/d = 0.5042$, the curves for reflection coefficients do not behave regularly. For example, in some range the curve for $\theta = 60°$ is higher than the others, and in another range that for $\theta = 75°$ surpasses it. Outside of this range, however, the steeper the discontinuity, the more the power will be reflected back, no matter how great t_2/d is. Also, we note that the curve for the coefficient of the first (second) mode is first lowered (raised) and then raised (lowered) as the tapered angle is increased.

V. Conclusions

The partial variation operator has been defined to establish a partial variational principle (PVP) for electromagnetic field problems. The problem of discontinuities in partially dielectric-filled waveguides has been tackled as an application of this PVP.

In this paper, the definition of reactions has been enlarged so that the variational equations in PVP can be used with more flexibility. With the choice for (p,q) in the definition of reaction as $(+,-)$ or $(-,+)$, the continuity

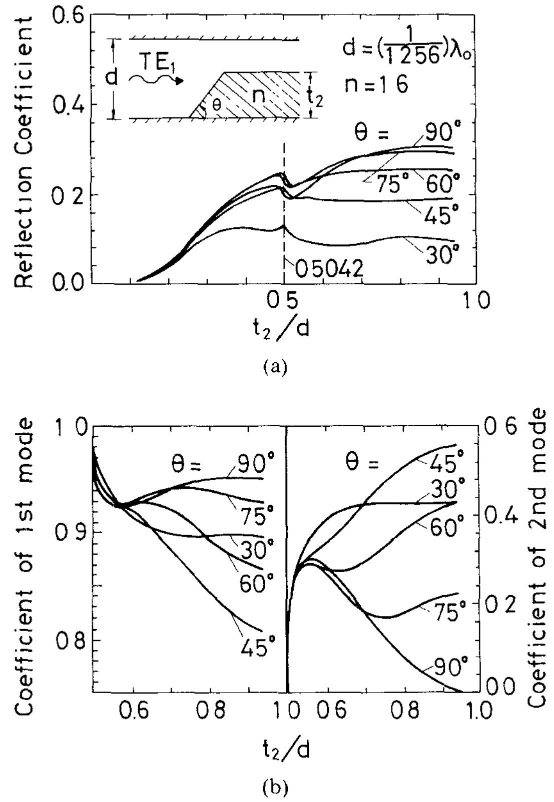

Fig. 7 Reflection and transmission coefficients for straight tapered-discontinuity problem with tapered angles as parameters.

conditions at each boundary need not be enforced. This flexibility may be useful in many electromagnetic problems such as the slab waveguide discontinuity problems and the open-type waveguide problems. Such problems are under investigation and will appear in the near future.

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References

[1] R. F. Harrington, *Time-Harmonic Electromagnetic Fields*. New York: McGraw-Hill, 1961.

[2] T. G. Hazel and A. Wexler, "Variational formulation of the Dirichlet boundary condition," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 385–390, 1972.

[3] Z. J. Csendes and P. Silvester, "Numerical solution of dielectric loaded waveguides: I—Finite-element analysis," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 1124–1131, 1970.

[4] Z. J. Csendes and P. Silvester, "Numerical solution of dielectric loaded waveguides: II—Modal approximation technique," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 504–509, 1971

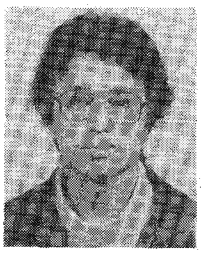
[5] P. Daly, "Hybrid-mode analysis of microstrip by finite-element methods," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 19–25, 1971.

[6] A. Konrad, "Vector variational formulation of electromagnetic fields in anisotropic media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 553–559, 1976.

[7] K. Morishita and N. Kumagai, "Unified approach to the derivation of variational expression for electromagnetic fields," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 34–40, 1977.

[8] K. Morishita and N. Kumagai, "Systematic derivation of variational expressions for electromagnetic and/or acoustic waves," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 684–689, 1978.

[9] G. Jeng and A. Wexler, "Self-adjoint variational formulation of problems having non-self-adjoint operators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 91–94, 1978.
[10] P. Silvester and M. S. Hsieh, "Finite-element solution of 2-dimensional exterior field problems," *Proc. Inst. Elec. Eng.*, vol. 118, pp. 1743–1747, 1971.
[11] B. H. McDonald and A. Wexler, "Finite-element solution of unbounded field problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 841–847, 1972.
[12] K. K. Mei, "Unimoment method of solving antenna and scattering problems," *IEEE Trans. Antennas Propagat.*, vol. AP-22, pp. 760–766, 1974.
[13] C. H. Chen and Y. W. Kiang, "A variational theory for wave propagation in a one-dimensional inhomogeneous medium," *IEEE Trans. Antennas Propagat.*, vol. AP-28, pp. 762–769, 1980.
[14] C. H. Chen and C. D. Lien, "The variational principle for non-self-adjoint electromagnetic problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 878–886, 1980.
[15] S. K. Jeng and C. H. Chen, "Variational finite element solution of electromagnetic wave propagation in a one-dimensional inhomogeneous anisotropic medium," *J. Appl. Phys.*, vol. 55, pp. 630–636, 1984.
[16] S. K. Jeng and C. H. Chen, "On variational electromagnetics: Theory and applications," *IEEE Trans. Antennas Propagat.*, vol. AP-32, pp. 902–907, 1984.
[17] R. B. Wu and C. H. Chen, "On the variational reaction theory for dielectric waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 477–483, 1985.
[18] R. B. Wu and C. H. Chen, "Variational reaction formulation of scattering problem for anisotropic dielectric cylinders," *IEEE Trans. Antennas Propagat.*, vol. AP-34, pp. 640–645, 1986.
[19] S. K. Jeng, R. B. Wu, and C. H. Chen, "Waves obliquely incident upon a stratified anisotropic slab: A variational reaction approach," *Radio Sci.*, vol. 21, pp. 681–688, 1986.
[20] V. H. Rumsey, "Reaction concept in electromagnetic theory," *Phys. Rev.*, vol. 94, pp. 1483–1491, June 1954.
[21] O. C. Zienkiewicz, *The Finite Element Method*. New York: McGraw-Hill, 1977.
[22] P. Hood, "Frontal solution program for unsymmetric matrices," *Int. J. Num. Meth. Eng.*, vol. 10, pp. 379–399, 1976.
[23] E. G. Royer and R. Mittra, "The diffraction of electromagnetic waves by dielectric steps in waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 273–279, 1972.
[24] S. W. Lee, W. R. Jones, and J. J. Campbell, "Convergence of numerical solutions of iris-type discontinuity problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 528–536, 1971.
[25] R. Mittra, T. Itoh, and T. S. Li, "Analytical and numerical studies of the relative convergence phenomenon arising in the solution of an integral equation by the moment method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 96–104, 1972.



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